ON r - DYNAMIC COLORING OF SOME GRAPHS
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ABSTRACT
The r-dynamic coloring of a graph H is a proper p-coloring of the vertices of the graph H so that for every vertex a ∈ V(H) has neighbors in at least min{r, d(a)} distinct classes of color. The least p which provides H an r-dynamic coloring with p colors is known as r-dynamic chromatic number of the graph H and it is denoted as χr(H). In this paper, we have attained the lower, upper bound and exact r-dynamic chromatic number for cocktail party graph Cp,s, s-barbell graph Ba,s, windmill graph Ws,q, book graph Bs and pencil graph Pc,s.

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Keywords: r-dynamic coloring; cocktail party graph Cp,s; s-barbell graph Ba,s; windmill graph Ws,q; book graph Bs; pencil graph Pc,s.

1. INTRODUCTION
Throughout we take into account simple, finite and connected graphs. Montgomery was the pioneer in dynamic coloring. Dynamic coloring [1, 2, 3, 5, 9, 10] of a graph is proper coloring of H so that each and every vertex a ∈ V(H) has neighbors in at least two different classes of color. And its generalized version is r-dynamic coloring. A mapping c : V(H) → Q, the set of colors with Q = |p|, is known as r-dynamic coloring if the following two rules holds:
1) c(a) ≠ c(z) for az ∈ E(G) and
2) |c(N(z))| ≥ min{r, d(z)}, for each and every z ∈ V(H) where N(z) denotes the set of neighbors of z, r is a positive integer and d(z) is the degree of the vertex z in H.

The first rule is an indication for proper coloring and the second rule is the r-adjacency condition. The r-dynamic chromatic number is the least p that allows H an r-dynamic coloring with p colors and it is denoted as χr(H). The r-dynamic chromatic number does not differ once r reaches the saturation value Δ. There are many open problems one among them was conjectured by Montgomery which states for regular graphs the result χr(H) ≤ χp(H) + 2. Graph coloring is one among the most challenging problems in mathematics and has many real-life applications.

2. PRELIMINARIES

[7] The Cocktail Party Graph Cp,s is a graph with s = 2q vertices aj, j = 1, 2, ..., 2q with aj non-adjacent to aq+j and adjacent to all other vertices.

[4] The s-Barbell Graph Ba,s is attained by connecting two copies of complete graph Ks by a bridge. Here, we are connecting the first two vertices of Ks by a bridge.

[12] The Windmill Graph Ws,q with q, s ≥ 2 is constructed by considering q copies of the complete graph Ks with a universal vertex (common vertex). When q = 2 and q = 3 i.e., Ws,2 and Ws,3 they are the star graph and friendship graph respectively.

[8] The Book Graph Bs is the Cartesian product of the star graph Ks and path P2 i.e., Ks × P2.

[6] For s ≥ 2, the Pencil Graph Pc,s is a graph with 2s + 2 vertices where the vertex set is {a0, b0 : q = 0, 1, ..., s} and edge set {aqaq+1, bqbq+1 : q = 1, 2, ..., s − 1} ∪ {a0a1, a0b1, b0as, b0bs} ∪ {aqaq : q = 0, 1, ..., s}.

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3. OBSERVATIONS

Lemma 3.1. \( \chi_r(H) \geq \min\{r, \Delta(H)\} + 1 \) is a lemma providing the lower bound for \( r \)-dynamic chromatic number found by Montgomery and Lai [9].

Note 3.2. We can observe easily from the graph \( C_p \) that there is a clique of order \( \frac{s}{2} = q \) hence \( \chi_r(C_p) \geq q \).

Note 3.3. From the definition of \( BA_s \) and \( W_s^q \) there is a maximal complete subgraph of order \( s \) hence \( \chi_r(BA_s) \geq s \) and \( \chi_r(W_s^q) \geq s \).

4. RESULTS

Lemma 4.1. For \( q \geq 2 \), the lower bound for \( r \)-dynamic chromatic number of cocktail party graph \( C_p \) is, \( \chi_r(C_p) \geq \{q, 1 \leq r \leq q - 1\} \). The cocktail party graph is \( C_p \) is regular graph with degree 2(q - 1). Then \( V(C_p) = \{a_j, j = 1, 2, \ldots, 2q\} \) and \( E(C_p) = \{a_ja_k: j, k = 1, 2, \ldots, 2q \text{ where } j \neq k = j + q\} \). Also, \( \Delta(C_p) = \Delta(C_p) = 2(q - 1) \). The order of \( C_p \) is \( |V(C_p)| = s = 2q \) and size is \( |E(C_p)| = s(q - 1) \).

By Note 3.2 we have \( \chi_{1sr \leq s-1}(C_p) \geq q \).

For \( q \leq r \leq \Delta(C_p) \), by Lemma 3.1 \( \chi_r(H) \geq \min\{r, \Delta(H)\} + 1 \). Therefore,

\[ \chi_{qsr \leq \Delta(C_p)}(C_p) \geq \min\{r, \Delta(C_p)\} + 1 = r + 1. \]

Theorem 4.2. For \( q \geq 2 \), the \( r \)-dynamic chromatic number of cocktail party graph \( C_p \) is, \( \chi_r(C_p) = \{q, 1 \leq r \leq q - 1\} \). Theorem 4.2. For \( q \geq 2 \), the \( r \)-dynamic chromatic number of cocktail party graph \( C_p \) is, \( \chi_r(C_p) \geq \{q, 1 \leq r \leq q - 1\} \).

Proof. We have two cases: \( 1 \leq r \leq q - 1 \) and \( q \leq r \leq \Delta(C_p) \) to consider.

Case 1: When \( 1 \leq r \leq q - 1 \).

By Lemma 4.1 we have the lower bound \( \chi_{1sr \leq s-1}(C_p) \geq q \). Consider the map \( c_1: V(C_p) \to \{1, 2, \ldots, q\} \) and coloring as follows:

- \( c_1(a_j) = j, j = 1, 2, \ldots, q \).
- \( c_1(a_j) = k, j = q + k \) and \( k = 1, 2, \ldots, q \) since \( d_j \) and \( d_{j+q} \) are non-adjacent.

By the above coloring \( \chi_{1sr \leq s-1}(C_p) \leq q \). Hence \( \chi_{1sr \leq s-1}(C_p) = q \).

Case 2: When \( q \leq r \leq \Delta(C_p) \).

By Lemma 4.1 we have \( \chi_{qsr \leq \Delta(C_p)}(C_p) \geq r + 1 \) but inorder to satisfy the \( r \)-adjacency condition we require \( 2q \) colors in total hence we the lower bound have \( \chi_{qsr \leq \Delta(C_p)}(C_p) \geq 2q \). The upper bound is given by the coloring below considering the mapping \( c_2: V(C_p) \to \{1, 2, \ldots, 2q\} \).

Theorem 4.4. For \( s \geq 2 \), the \( r \)-dynamic chromatic number of barbell graph \( BA_s \) is,

\[ \chi_r(BA_s) \geq \{s, 1 \leq r \leq s - 1\} \]

Theorem 4.4. For \( s \geq 2 \), the \( r \)-dynamic chromatic number of barbell graph \( BA_s \) is, \( \chi_r(BA_s) \geq \{s, 1 \leq r \leq s - 1\} \).

Proof. We have two cases:

Case 1: When \( 1 \leq r \leq s - 1 \).

By Lemma 4.3 we have \( \chi_{1sr \leq s-1}(BA_s) \geq s \). The coloring is provided by the mapping \( c_3: V(BA_s) \to \{1, 2, \ldots, s\} \) as follows:

- \( c_3(a_1, a_2, \ldots, a_s) = \{1, 2, \ldots, s\} \).
- \( c_3(b_1, b_2, \ldots, b_s) = \{2, 3, \ldots, s, 1\} \).

By the above coloring \( \chi_{1sr \leq s-1}(BA_s) \leq s \). Hence \( \chi_{1sr \leq s-1}(BA_s) = s \).
Case 2: When \( r = \Delta(Ba_s) \).
By Lemma 4.3 we have the lower bound \( \chi_r(\Delta(Ba_s)) \geq s + 1 \). The upper bound is given by the coloring below considering the mapping \( c_4 : V(Ba_s) \rightarrow \{1, \ldots, s + 1\} \).
\( c_4(a_1, a_2, \ldots, a_s) = \{1, 2, \ldots, s\} \) and \( c_4(b_1) = s + 1 \) and \( c_3(b_2, b_3, \ldots, b_s) = \{2, 3, \ldots, s\} \).
\( \chi_r(\Delta(Ba_s)) \leq s + 1 \).

Therefore, \( \chi_r(\Delta(Ba_s)) = s + 1 \).

**Lemma 4.5.** For \( q, s \geq 2 \), the lower bound for \( r \)-dynamic chromatic number of windmill graph \( W^q_s \) is, \( \chi_r(W^q_s) \geq \begin{cases} s, & 1 \leq r \leq s - 1 \\ r + 1, & s \leq r \leq \Delta(W^q_s) \end{cases} \)

**Proof.** The vertex set of windmill graph \( V(W^q_s) = \{a_0\} \cup \{a_{j,1}, a_{j,2}, \ldots, a_{j,s-1} : 1 \leq j \leq q\} \) where \( a_0 \) is universal vertex adjacent to all other vertices \( \{a_{j,k} : 1 \leq j \leq q \text{ and } 1 \leq k \leq s - 1\} \).

Edge set is \( E(W^q_s) = \{a_{j,k}a_{i,j} : k \neq i \text{ and } 1 \leq j \leq q, 1 \leq k, i \leq s - 1\} \) \cup \{a_0a_{j,k} : 1 \leq j \leq q \text{ and } 1 \leq k \leq s - 1\} \). Also, \( \delta(W^q_s) = s - 1 \) and \( \Delta(W^q_s) = d(a_0) = q(s - 1) \). And, order of \( W^q_s \) is \( |V(W^q_s)| = q(s - 1) + 1 \) and size is \( |E(W^q_s)| = \frac{qs(s-1)}{2} \).

By Note 3.3 we have \( \chi_{1s}rs1s-1(W^q_s) \geq s \).
For \( s \leq r \leq \Delta(W^q_s) \), by Lemma 3.1 \( \chi_r(H) \geq \min\{r, \Delta(H)\} + 1 \).

Therefore,
\( \chi_{srs1s}r1s-1s(W^q_s) \geq \min\{r, \Delta(W^q_s)\} + 1 \).
\( r + 1 \).

**Theorem 4.6.** For \( q, s \geq 2 \), the \( r \)-dynamic chromatic number of windmill graph \( W^q_s \) is,
\( \chi_r(W^q_s) = \begin{cases} s, & 1 \leq r \leq s - 1 \\ r + 1, & s \leq r \leq \Delta(W^q_s) \end{cases} \)

**Proof.** We have two cases: \( 1 \leq r \leq s - 1 \) and \( s \leq r \leq \Delta(W^q_s) \) to consider.

**Case 1:** When \( 1 \leq r \leq s - 1 \).
By Lemma 4.5 we have the lower bound \( \chi_{1s}rs1s-1s(W^q_s) \geq s \). The coloring is provided by the map \( c_5 : V(W^q_s) \rightarrow \{1,2,\ldots,s\} \) as follows:
\( c_5(a_0) = 1 \).
\( c_5(a_{j,1}, a_{j,2}, \ldots, a_{j,s-1}) = \{2,3,\ldots,s\} \) for \( 1 \leq j \leq q \).

By the above coloring \( \chi_{1s}rs1s-1s(W^q_s) \leq s \). Hence \( \chi_{1s}rs1s-1s(W^q_s) = s \).

**Case 2:** When \( s \leq r \leq \Delta(W^q_s) \).
By Lemma 4.5 we have \( \chi_{srs1s}r1s-1s(W^q_s) \geq r + 1 \). The upper bound is given by the coloring below considering the mapping \( c : V(W^q_s) \rightarrow \{1,2,\ldots,r + 1\} \) for different stages of \( r \).
\( c(a_0) = 1 \) for all cases of \( r \).
When \( r = s \),
\( c(a_{1,1}, a_{1,2}, \ldots, a_{1,s-1}) = \{2,3,\ldots,s\} \)
\( c(a_{2,1}, a_{2,2}, \ldots, a_{2,s-1}) = \{s + 1,3,4,\ldots,s\} \)
\( c(a_{j,1}, a_{j,2}, \ldots, a_{j,s-1}) = \{2,3,\ldots,s\} \) for \( 3 \leq j \leq q \).

Hence \( \chi_{r=s}(W^q_s) \leq s + 1 \).

When \( r = s + 1 \),
\( c(a_{2,1}, a_{2,2}, \ldots, a_{2,s-1}) = \{s + 1, s + 2,4,\ldots,s\} \)
\( c(a_{j,1}, a_{j,2}, \ldots, a_{j,s-1}) = \{2,3,\ldots,s\} \) for \( 1 \leq j \leq q \) and \( j \neq 2 \).

Hence \( \chi_{r=s+1}(W^q_s) \leq s + 2 \).

Proceeding like this at each stage of \( r \) we introduce the color \( r+1 \) to the next vertex in the list till \( a_{j,s-1} \).

Hence \( \chi_{srs1s}r1s-1s(W^q_s) \leq r + 1 \).

Therefore, \( \chi_{srs1s}r1s-1s(W^q_s) = r + 1 \).

**Lemma 4.7.** For \( s \geq 2 \), the lower bound for \( r \)-dynamic chromatic number of book graph \( B_s \) is,
\( \chi_r(B_s) \geq \begin{cases} 2, & r = 1 \\ r + 1, & 2 \leq r \leq \Delta(B_s) \end{cases} \)

**Proof.** The vertex set of book graph \( V(B_s) = \{b_{1,j}, b_{2,j}, x, y : 1 \leq j \leq s\} \) and edge set \( E(B_s) = \{xb_{1,j}, yb_{2,j}, b_{1,j}b_{2,j} : 1 \leq j \leq s\} \).

Also, \( \delta(B_s) = 2 \) and \( \Delta(B_s) = d(x) = d(y) = s \). And, order of \( B_s \) is \( |V(B_s)| = 2s + 2 \) and size is \( |E(B_s)| = 3s \).

Since the maximal complete subgraph is of order 2 we have \( \chi_{r=1}(Ba_s) \geq 2 \).
For \( 2 \leq r \leq \Delta(B_s) \), by Lemma 3.1 \( \chi_r(H) \geq \min\{r, \Delta(H)\} + 1 \).

Therefore, \( \chi_{srs1s}r1s-1s(B_s) \geq \min\{r, \Delta(Ba_s)\} + 1 \).
\( r + 1 \).
Theorem 4.8. For \( s \geq 2 \), the \( r \) -dynamic chromatic number of book graph \( B_s \) is, 
\[
\chi_r(B_s) = \begin{cases} 
2, & r = 1 \\
4, & r = 2, 3 \\
r + 1, & 4 \leq r \leq \Delta(B_s) 
\end{cases}
\]
Proof. We have two cases: \( r = 1 \, \text{and} \, r = 2, 3 \) and 
\[
4 \leq r \leq \Delta(B_s). 
\]
Case 1: When \( r = 1 \).
By Lemma 4.7 we have the lower bound 
\[
\chi_1(B_{4s}) \geq 2. 
\]
The coloring is provided by the map 
\[
c_6: V(B_s) \to \{1,2\} \text{ as follows:} 
\]
\[
c_6(x) = 1 \quad \text{and} \quad c_6(y) = 2. 
\]
\[
c_6(b_{1,j}) = 2 \quad \text{and} \quad c_6(b_{2,j}) = 1 \quad \text{for} \quad 1 \leq j \leq s. 
\]
By the above coloring \( \chi_1(B_s) \leq 2. \) Hence 
\[
\chi_1(B_s) = 2. 
\]
Case 2: When \( r = 2, 3 \).
By Lemma 4.7 we have the lower bound 
\[
\chi_{r=2}(B_s) \geq r + 1 = 3. 
\]
But since there is presence of \( C_4 \) in \( B_s \) which leads to the need of an extra color when \( r = 2 \). So, the lower bound becomes \( \chi_{r=2}(B_s) \geq 4 \) and by the same lemma we have the lower bound \( \chi_{r=3}(B_s) \geq r + 1 = 4. \)
Now we assign coloring by the mapping 
\[
c_7: V(B_s) \to \{1,2,3,4\} \text{ as follows:} 
\]
\[
c_7(x) = 1 \quad \text{and} \quad c_7(y) = 2. 
\]
\[
c_7(b_{1,j}) = \begin{cases} 
3, & j \text{ odd} \\
4, & j \text{ even} 
\end{cases} \quad \text{and} \quad c_7(b_{2,j}) = 
\begin{cases} 
4, & j \text{ odd} \\
3, & j \text{ even} 
\end{cases} 
\]
For \( r = 3 \) the coloring provided above is sufficient.
So, we have the upper bound \( \chi_{r=3}(B_s) \leq 4. \)
Therefore, \( \chi_{r=2,3}(B_s) = 4. \)
Case 3: When \( 4 \leq r \leq \Delta(B_s). \)
By Lemma 4.7 we have the lower bound 
\[
\chi_{4s \leq \Delta(B_s)}(B_s) \geq r + 1. 
\]
Consider the mapping 
\[
c : V(B_s) \to \{1,2,\ldots, r + 1\} \text{ which gives the} 
\]
coloring for the vertices.
\[
c(x) = 1 \quad \text{and} \quad c(y) = 2. 
\]
\[
c(b_{1,1}, b_{1,2}, \ldots, b_{1,s}) = \{3,4,\ldots, r + 1, 3,4,\ldots\} \quad \text{for} \quad s-(r-1) \text{ terms} 
\]
\[
c(b_{2,1}, b_{2,2}, \ldots, b_{2,s}) = \{4,\ldots, r + 1, 3,4,\ldots\} \quad \text{for} \quad s-(r-1) \text{ terms} 
\]
Hence \( \chi_{4s \leq \Delta(B_s)}(B_s) \leq r + 1. \) Therefore, we have 
\[
\chi_{4s \leq \Delta(B_s)}(B_s) = r + 1. 
\]
Lemma 4.9. For \( s \geq 2 \), the lower bound for \( r \) -dynamic chromatic number of pencil graph \( P_{Cs} \) is, 
\[
\chi_r(P_{Cs}) \geq \begin{cases} 
3, & r = 1, 2 \\
4, & r \geq \Delta(P_{Cs}) 
\end{cases} 
\]
Proof. The pencil graph \( P_{Cs} \) is a regular graph with degree 3. The vertex set is defined as \( \{a_q, b_q : q = 0,1,\ldots, s\} \) and edge set \( \{a_qa_{q+1}, b_qb_{q+1} : q = 1,2,\ldots, s - 1\} \cup \{a_0a_1, a_0b_1, b_0a_s, b_0b_s\} \cup \{a_qb_q : q = 0,1,\ldots, s\}. \) Also, 
\[
\delta(P_{Cs}) = \Delta(P_{Cs}) = 3. \text{ And, order of} \ P_{Cs} \text{ is} 
\]
\[
|V(P_{Cs})| = 2s + 2 \text{ and size is} \ |E(P_{Cs})| = 3(s + 1). 
\]
There is a clique of order 3 we have \( \chi_{r=1,2}(P_{Cs}) \geq 3. \)
For \( r \geq \Delta(P_{Cs}), \) by Lemma 3.1 
\[
\chi_r(H) \geq \min\{r,\Delta(H)\} + 1. 
\]
Therefore, \( \chi_{r=\Delta(P_{Cs})}(B_s) \geq \min\{r,\Delta(P_{Cs})\} + 1 = \Delta(P_{Cs}) + 1 = 4. \)
Theorem 4.10. For \( s \geq 2 \), the \( r \) -dynamic chromatic number of pencil graph \( P_{Cs} \) is, 
\[
\chi_r(P_{Cs}) = \begin{cases} 
3, & r = 1, 2 \quad s \equiv 0,2 (mod 3), \\
r = 1 \text{ and} \ s \equiv 1 (mod 3) \\
4, & r = 2 \text{ and} \ s \equiv 1 (mod 3) \\
5, & r = 3 \text{ and} \ s \equiv 0 (mod 4) \\
6, & r = 3 \text{ and otherwise} 
\end{cases} 
\]
Proof. We have four cases to consider here.
Case 1: When \( r = 1, 2 \text{ and} \ s \equiv 0,2 (mod 3), r = 1 \text{ and} \ s \equiv 1 (mod 3). \)
Subcase 1: When \( r = 1, 2 \text{ and} \ s \equiv 0,2 (mod 3) \)
By Lemma 4.9 we have the lower bound 
\[
\chi_{r=1,2}(P_{Cs}) \geq 3. \text{ For upper bound consider the map} \ c_6: V(P_{Cs}) \to \{1,2,3\}. 
\]
\[
c_6(a_0) = 3 
\]
\[
c_6(a_1, a_2, \ldots, a_s) = \{1,2,3,1,2,3, \ldots, 1,2,3\} \text{ when} \ s \equiv 0 (mod 3) 
\]
\[
c_6(a_1, a_2, \ldots, a_s) = \{1,2,3,1,2,3, \ldots, 1,2\} \text{ when} \ s \equiv 2 (mod 3) 
\]
\[
c_6(b_1, b_2, \ldots, b_s) = \{3,2,1,3,2,1, \ldots, 3,2,1\} \text{ when} \ s \equiv 0 (mod 3) 
\]
c_8(b_1, b_2, \ldots, b_5) = \{3, 2, 1, 3, 2, 1, \ldots, 3, 2\}

when \( s \equiv 2(\text{mod } 3) \\
c_8(b_0) = \begin{cases} 
2, & \text{when } s \equiv 0(\text{mod } 3) \\
1, & \text{when } s \equiv 2(\text{mod } 3) 
\end{cases}

Hence \( \chi_{r=1,2}(P_{C_5}) \leq 3 \) and therefore \( \chi_{r=1,2}(P_{C_5}) = 3 \) when \( s \equiv 0, 2(\text{mod } 3) \).

**Subcase 2:** When \( r = 1 \) and \( s \equiv 1(\text{mod } 3) \).

By Lemma 4.9 we have the lower bound \( \chi_{r=1}(P_{C_5}) \geq 3 \). Consider the map \( c_8: V(P_{C_5}) \rightarrow \{1, 2, 3\} \).

\( c_8(a_1, a_2, \ldots, a_5) = \{1, 2, 1, 2, \ldots, 1\} \) when \( s \) is even

\( c_8(a_1, a_2, \ldots, a_5) = \{1, 2, 1, 2, \ldots, 1\} \) when \( s \) is odd

\( c_8(b_1, b_2, \ldots, b_5) = \{2, 1, 2, 1, \ldots, 2, 3\} \) when \( s \) is even

\( c_8(b_1, b_2, \ldots, b_5) = \{2, 1, 2, 1, \ldots, 2, 3\} \) when \( s \) is odd

\( c_8(a_0) = 3 \) and \( c_8(b_0) = \begin{cases} 
1, & \text{when } s \text{ is even} \\
2, & \text{when } s \text{ is odd} 
\end{cases} 

Hence \( \chi_{r=1}(P_{C_5}) \leq 3 \) and therefore \( \chi_{r=1}(P_{C_5}) = 3 \) when \( s \equiv 1(\text{mod } 3) \).

**Case 2:** When \( r = 2 \) and \( s \equiv 1(\text{mod } 3) \).

By Lemma 4.9 we have the lower bound \( \chi_{r=2}(P_{C_5}) \geq 3 \). But we need an extra color when \( s \equiv 1(\text{mod } 3) \) to satisfy \( r \)-adjacency condition. So, the lower bound transforms to \( \chi_{r=2}(P_{C_5}) \geq 4 \).

For upper bound assign the following coloring provided with the mapping \( c_9: V(P_{C_5}) \rightarrow \{1, 2, 3, 4\} \).

\( c_9(a_0) = 3 \) and \( c_9(b_0) = 4 \)

\( c_9(a_1, a_2, \ldots, a_5) = \{1, 2, 3, 1, 2, 3, \ldots\} \)

\( c_9(b_1, b_2, \ldots, b_5) = \{2, 3, 1, 2, 3, 1, \ldots\} \)

Hence \( \chi_{r=2}(P_{C_5}) \leq 4 \). Therefore, \( \chi_{r=2}(P_{C_5}) = 4 \) when \( s \equiv 1(\text{mod } 3) \).

**Case 3:** When \( r = 3 \) and \( s \equiv 0(\text{mod } 4) \).

By Lemma 4.9 we have the lower bound \( \chi_{r=3}(P_{C_5}) \geq r + 1 = 4 \). But inorder to \( r \)-adjacency condition we need an extra color and hence the lower bound transforms to \( \chi_{r=3}(P_{C_5}) \geq 5 \).

Assign the coloring by the map \( c_{10}: V(P_{C_5}) \rightarrow \{1, 2, 3, 4, 5\} \).

For \( 1 \leq q \leq s \), c_{10}(a_q) = \begin{cases} 
1, & \text{when } q \equiv 1(\text{mod } 4) \\
4, & \text{when } q \equiv 2(\text{mod } 4) \\
3, & \text{when } q \equiv 3(\text{mod } 4) \\
2, & \text{when } q \equiv 0(\text{mod } 4) \\
1, & \text{when } q \equiv 1(\text{mod } 4) \\
5, & \text{when } q \equiv 2(\text{mod } 4) \\
4, & \text{when } q \equiv 0(\text{mod } 4) 
\end{cases} 

\( c_{10}(b_0) = \begin{cases} 
3, & \text{when } s \equiv 0(\text{mod } 4) \\
6, & \text{otherwise} \end{cases} 

Thus, the upper bound is \( \chi_{r=3}(P_{C_5}) \leq 5 \). Therefore, \( \chi_{r=3}(P_{C_5}) = 5 \).

**Case 4:** When \( r = 3 \) and otherwise.

By Lemma 4.9 we have the lower bound \( \chi_{r=3}(P_{C_5}) \geq 4 \). But inorder to \( r \)-adjacency condition we are forced to introduce two new colors and hence the lower bound \( \chi_{r=3}(P_{C_5}) \geq 6 \).

Assign the coloring by the map \( c_{11}: V(P_{C_5}) \rightarrow \{1, 2, \ldots, 6\} \).

\( c_{11}(a_1, a_2, \ldots, a_5) = \{1, 4, 2, 5, 1, 4, 2, 5, \ldots, 1\} \) when \( s \equiv 1(\text{mod } 4) \)

\( c_{11}(a_1, a_2, \ldots, a_5) = \{1, 4, 2, 5, 1, 4, 2, 5, \ldots, 1\} \) when \( s \equiv 2(\text{mod } 4) \)

\( c_{11}(a_1, a_2, \ldots, a_5) = \{1, 4, 2, 5, 1, 4, 2, 5, \ldots, 1, 4\} \) when \( s \equiv 3(\text{mod } 4) \)

\( c_{11}(b_1, b_2, \ldots, b_5) = \{2, 5, 1, 4, 2, 5, 1, 4, \ldots, 2\} \) when \( s \equiv 1(\text{mod } 4) \)

\( c_{11}(b_1, b_2, \ldots, b_5) = \{2, 5, 1, 4, 2, 5, 1, 4, \ldots, 2, 5\} \) when \( s \equiv 2(\text{mod } 4) \)

\( c_{11}(b_1, b_2, \ldots, b_5) = \{2, 5, 1, 4, 2, 5, 1, 4, \ldots, 2, 5, 1\} \) when \( s \equiv 3(\text{mod } 4) \)

\( c_{11}(a_0) = 3 \) and \( c_{11}(b_0) = 6 \).

Thus, the upper bound is \( \chi_{r=3}(P_{C_5}) \leq 6 \). Therefore, \( \chi_{r=3}(P_{C_5}) = 6 \), otherwise i.e., when \( s \equiv 1, 2, 3(\text{mod } 4) \).
REFERENCES


