Kongunadu Research Journal 12 (2): 102-107-, 2025 ISSN 2349-2694, All Rights Reserved,
Publisher: Kongunadu Arts and Science College, Coimbatore. https://krjournal.com/index.php/krj/article/view/740

DOI:10.26524/krj.2025.25

RESEARCH ARTICLE

A STUDY ON NANO *AG*P*-REGULAR AND NANO *AG*P*-NORMAL SPACES

Adinatha C. Upadhya[1] and Mamata M. K[2]

[1]Associate Professor, Department of Mathematics, S.R.F.G.Constituent College, Belagavi-590017. Karnataka. India.
[2]Department of Mathematics, G.S.Science College, Belagavi-590011. Karnataka. India.
Email: adinatha06@gmail.com

Abstract

In recent years, nano topology has emerged as an important generalization of classical topology, providing a useful framework for dealing with uncertainty and approximation. In this paper, we introduce and investigate two new classes of separation axioms in nano topological spaces, namely nano αg^*p -regular and nano αg^*p -normal spaces. These spaces are defined by utilizing the properties of nano αg^*p -open sets.

Keywords: Nano Open Sets, Nano αg^*p-Regular Space, Nano αg^*p-Normal Space, Nano Continuous.

1. Introduction

Nano topology, introduced by Thivagar [7] based on rough set theory proposed by Pawlak [4], generalizes classical topology by incorporating lower and upper approximations and boundary regions induced by an equivalence relation on a finite universe. This framework provides an effective tool for handling uncertainty in topological structures. Subsequently, Thivagar and Richard [8,9] developed fundamental concepts such as nano interior, nano closure, nano continuity, and several classes of nano open sets.

Generalized open and closed sets play a crucial role in the study of separation axioms in topology. Motivated by classical generalized sets such as α-open and pre-open sets introduced by Reilly and Vamanamurty [5], several authors extended these notions to nano topological spaces. Bhuvaneswari and Mythili Ganapriya [1] introduced nano generalized closed sets and studied their properties, leading to further investigations into generalized separation axioms. Regularity and normality in nano topological spaces were later examined using various generalized open set structures [2,6,10].

Recent studies have focused on generalized forms of regularity and normality in nano topology. Upadhya [11] introduced quasi nano p-normal spaces, while Upadhya and Mamata [12] investigated nano p-regular and nano p-normal spaces. More recently, nano $g\alpha$-normal spaces were studied in [13], emphasizing the importance of generalized nano closed sets in refining separation properties.

Motivated by these developments, this paper introduces and studies nano αg^*p-regular and nano αg^*p-normal spaces, defined via nano αg^*p-open and nano αg^*p-closed sets. Several characterizations are obtained, relationships with existing nano regular and nano normal spaces are established, and preservation properties under subspaces and mappings are examined.

2. Preliminaries

In this section, we review the fundamental definitions and results of nano topological spaces Throughout this paper, U denotes a non-empty finite universe and R an equivalence relation on U. The pair $(U, R)$is called an approximation space.

Definition 2.1[4]: Let $X \subseteq U$. The lower approximation, upper approximation, and boundary region of $X$with respect to $R$are defined as $\underline{R}(X) = \{x \in U: [x]_R \subseteq X\}$, $\bar{R}(X) = \{x \in U: [x]_R \cap X \neq \emptyset\}$ and $B_R(X) = \bar{R}(X) \setminus \underline{R}(X)$, where $[x]_R$ denotes the equivalence class of $x$under R.

Definition 2.2[7]: The collection $\tau_R(X) = \{\emptyset, U, \underline{R}(X), \bar{R}(X), B_R(X)\}$ forms a topology on U, called the nano topology induced by X. The triple $(U, \tau_R(X))$ is called a nano topological space. Elements of $\tau_R(X)$are called nano open sets, and their complements are called nano closed sets.

Definition 2.3[8]: Let $A \subseteq U$. The nano closure of A, denoted by $\mathrm{Cl}(A)$, is the intersection of all nano closed sets containing A. The nano interior of A, denoted by $\mathrm{Int}(A)$, is the union of all nano open sets contained in A.

Definition 2.4[5,9]: A subset $A$of a nano topological space $U$is said to be:

(i) nano semi-open if $A \subseteq NCl(NInt(A))$

(ii) nano pre-open if $A \subseteq N\text{Int}(N\text{Cl}(A))$,

(iii) nano α-open if $A \subseteq N\text{Int}(N\text{Cl}(N\text{Int}(A)))$.

The complements of these sets are called nano pre-closed and nano α-closed sets, respectively.

Definition 2.5: A subset $A \subseteq U$is said to be nano αg^*p-closed if $\text{pCl}(A) \subseteq V$ whenever $A \subseteq V$and $V$is a nano αg^*p-open set. The complement of a nano αg^*p-closed set is called a nano αg^*p-open set.

Definition 2.6: For any subset $A \subseteq U$, the nano αg^*p -closure of A, denoted by $\text{pCl}(A)$, is the intersection of all nano αg^*p-closed sets containing A.

Definition 2.7: Let $f: (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a function. Then $f$is said to be:

(i) nano continuous if the inverse image of every nano open set in $V$is nano open in U,

(ii) nano αg^*p-irresolute if the inverse image of every nano αg^*p-closed set in $V$is nano αg^*p-closed in U,

(iii) almost nano αg^*p-irresolute if the inverse image of every nano αg^*p-closed set is nano closed,

(iv) nano open if the image of every nano open set in $U$is nano open in V.

3. Nano *αg*p*-Regular Spaces

Definition 3.1: A nano topological space U is said to be nano αg*p- regular if for every nano αg*p-closed set F and a point u∉F, there exist disjoint nano - open sets V, W ∈ U such that u ∈ V and F ⊂ W.

Theorem 3.2: Every nano regular space is nano αg*p – regular space.

Proof Let Ube a nano regular space. By definition, for every point $x \in U$and for every nano closed set Fin $U$with $x \notin F$, there exist two disjoint nano open sets $V$and Wsuch that $x \in V$and $F \subseteq W$. Now, let Fbe a nano αg^*p-closed set in $U$and let $x \in U \setminus F$. Since every nano αg^*p-closed set is also a nano closed set (by definition), $F$is a nano closed set. Applying the nano regularity of U, there exist two disjoint nano open sets V and W such that $x \in V$ and $F \subseteq W$. Therefore, $U$is nano αg^*p-regular.

Remark 3.3: The reverse implication of the above Theorem need not be true can be seen from the following example.

Consider the nano topological space $U = \{a, b, c\}$with $\tau_R = \{\emptyset, \{a\}, \{a, b\}, U\}$. The nano αg^*p-closed sets can be taken as $\{c\}, \{b, c\}, U$. For any point $x \in U$and any nano αg^*p-closed set Fnot containing x, we can find disjoint nano open sets separating them, showing that $U$is nano αg^*p-regular. However, $U$is not nano regular because, for the nano closed set $F = \{b, c\}$and $x = a$, no two disjoint nano open sets exist separating x and F. Hence, this space demonstrates that a nano αg^*p-regular space need not be nano regular, so the converse of the theorem fails.

Theorem 3.4: In a nano topological space U, the following conditions are equivalent: (i) $U$is a nano αg^*p-regular space.

(ii) For each point $x \in U$and for each nano αg^*p-open set Acontaining x, there exists a nano open set Vsuch that $x \in V \subseteq \text{Cl}(V) \subseteq A$.

Proof: Assume first that $U$is a nano αg^*p-regular space. Let $x$be any point of $U$and let Abe a nano αg^*p-open set containing x. Then the complement $F = U \setminus A$is a nano αg^*p-closed set and $x \notin F$. By the definition of nano αg^*p-regularity, there exist two disjoint nano open sets $V$and Wsuch that $x \in V$and $F \subseteq W$. Since $V$and $W$are disjoint, $V$is contained in the complement of W. As W is nano open, its complement is nano closed, and hence the nano closure of $V$is contained in the complement of W. Further, since $F \subseteq W$, the complement of W is contained in the complement of F, which is precisely the set A. Therefore, $x \in V \subseteq \text{Cl}(V) \subseteq A$.

Conversely, suppose that for each point $x \in U$and each nano αg^*p-open set Acontaining x, there exists a nano open set Vsuch that $x \in V$and $\text{Cl}(V) \subseteq A$. Let Fbe any nano αg^*p-closed set and let $x \in U$with $x \notin F$. Then the complement $A = U \setminus F$is a nano αg^*p-open set containing x. By the assumption, there exists a nano open set Vcontaining $x$such that its nano closure is contained in $U \setminus F$. Let $W = U \setminus \text{Cl}(V)$. Since the nano closure of a set is nano closed, the set W is nano open. Moreover, the inclusion $\text{Cl}(V) \subseteq U \setminus F$implies that $F \subseteq W$. Clearly, the nano open sets $V$and $W$are disjoint, with $x \in V$and $F \subseteq W$. Hence, $U$is a nano αg^*p-regular space.

Theorem 3.5: A nano topological space $U$is nano αg^*p-regular if and only if for each nano αg^*p-closed set $F$of $U$and for each point $x \in U \setminus F$, there exist nano open sets $V$and Win U such that $x \in V, F \subseteq W$,and$\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$.

Proof: Necessity: Assume that $U$is a nano αg^*p-regular space. Let Fbe any nano αg^*p-closed set of $U$and let $x \in U \setminus F$. By the definition of nano αg^*p-regularity, there exist two disjoint nano open sets $V$and Win Usuch that $x \in V$and $F \subseteq W$. Since $V$and $W$are disjoint, the closure of $V$is contained in the complement of W, and the closure of $W$is contained in the complement of V. Consequently, the nano closures of V and W are also disjoint, that is, $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$.

Sufficiency: Conversely, assume that for each nano αg^*p-closed set F of U and each point $x \in U \setminus F$, there exist nano open sets V and W in U such that $x \in V$, $F \subseteq W$, and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since $\mathrm{Cl}(V)$ and $\mathrm{Cl}(W)$ are disjoint nano closed sets, the open sets V and W themselves must also be disjoint. Thus, we have two disjoint nano open sets separating the point x and the nano αg^*p-closed set F. By definition, this implies that the space U is nano αg^*p-regular.

Theorem 3.6: Every subspace of a nano αg^*p-regular space is also a nano αg^*p-regular space.

Proof: Let U be a nano αg^*p-regular space and let Y be any subspace of U. We shall show that Y, with the relative nano topology, is nano αg^*p-regular. Let F be a nano αg^*p-closed set in the subspace Y and let $x \in Y \setminus F$. Then there exists a nano αg^*p-closed set F_1 in U such that $F = Y \cap F_1$. Since $x \notin F$, we have $x \notin F_1$. Because U is nano αg^*p-regular, there exist nano open sets V and W in U such that $x \in V$, $F_1 \subseteq W$, and $V \cap W = \emptyset$. Define $V_Y = V \cap Y$ and $W_Y = W \cap Y$. Then V_Y and W_Y are nano open sets in the subspace Y, $x \in V_Y$, and $F \subseteq W_Y$. Moreover, since $V \cap W = \emptyset$, it follows that $V_Y \cap W_Y = \emptyset$. Thus, for the point x and the nano αg^*p-closed set F in Y, we have found disjoint nano open sets in Y separating them. Therefore, the subspace Y is nano αg^*p-regular. Hence, every subspace of a nano αg^*p-regular space is itself nano αg^*p-regular.

Theorem 3.7: For a nano topological space U, the following conditions are equivalent:

i. U is a nano αg^*p-regular space.
ii. For each point $x \in U$ and for each nano αg^*p-open set V of U containing x, there exists a nano open set W in U such that $x \in W \subseteq \mathrm{Cl}(W) \subseteq V$.
iii. For each point $x \in U$ and for each nano αg^*p-closed set A of U not containing x, there exists a nano open set W in U such that $\mathrm{Cl}(W) \cap A = \emptyset$.

Proof: Let U be a nano topological space. We prove that conditions (i), (ii), and (iii) are equivalent.

(i) ⇒ (ii): Assume that U is a nano αg^*p-regular space. Let $x \in U$ and let V be a nano αg^*p-open set of U containing x. Then the complement $A = U \setminus V$ is a nano αg^*p-closed set and does not contain x. By the definition of nano αg^*p-regularity, there exist two disjoint nano open sets W and G in U such that $x \in W$ and $A \subseteq G$. Since W and G are disjoint, the set W is contained in the complement of G. As G is nano open, its complement is nano closed, and hence the nano closure of W is also contained in the complement of G. Moreover, since $A \subseteq G$, the complement of G is contained in the complement of A, which is precisely the set V. Hence, $x \in W \subseteq \mathrm{Cl}(W) \subseteq V$.

(ii) ⇒ (iii): Assume that for each point $x \in U$ and each nano αg^*p-open set V containing x, there exists a nano open set W such that $x \in W \subseteq \mathrm{Cl}(W) \subseteq V$. Let A be any nano αg^*p-closed set of U not containing x. Then the complement $V = U \setminus A$ is a nano αg^*p-open set containing x. By the assumption, there exists a nano open set W such that $x \in W$ and $\mathrm{Cl}(W) \subseteq V$. Consequently, $\mathrm{Cl}(W)$ is disjoint from A, that is, $\mathrm{Cl}(W) \cap A = \emptyset$.

(iii) ⇒ (i): Assume that for each point $x \in U$ and each nano αg^*p-closed set A not containing x, there exists a nano open set W such that $\mathrm{Cl}(W) \cap A = \emptyset$. Let F be any nano αg^*p-closed set of U and let $x \in U \setminus F$. By the assumption, there exists a nano open set W containing x such that $\mathrm{Cl}(W) \cap F = \emptyset$. Define $G = U \setminus \mathrm{Cl}(W)$. Since the nano closure of a set is nano closed, the set G is nano open. Moreover, $\mathrm{Cl}(W) \cap F = \emptyset$ implies that $F \subseteq G$. Clearly, the nano open sets W and G are disjoint, with $x \in W$ and $F \subseteq G$. Hence, U is a nano αg^*p-regular space.

Theorem 3.8: Let $f: (U, \tau_R(X)) \longrightarrow (V, \tau_R(Y))$ be a bijective function. If f is nano αg^*p-irresolute and nano open, and if $(U, \tau_R(X))$ is a nano αg^*p-regular space, then $(V, \tau_R(Y))$ is also a nano αg^*p-regular space.

Proof Let $f: (U, \tau_R(X)) \longrightarrow (V, \tau_R(Y))$ be a bijective function which is nano αg^*p-irresolute and nano open, and suppose that $(U, \tau_R(X))$ is a nano αg^*p-regular space. We shall prove that $(V, \tau_R(Y))$ is nano αg^*p-regular. Let F be any nano αg^*p-closed set in $(V, \tau_R(Y))$ and let $y \in V \setminus F$. Since f is bijective, there exists a unique point $x \in U$ such that $f(x) = y$. Because f is nano αg^*p-irresolute, the inverse image $f^{-1}(F)$ is a nano αg^*p-closed set in $(U, \tau_R(X))$. Clearly, $x \notin f^{-1}(F)$. As $(U, \tau_R(X))$ is nano αg^*p-regular, there exist two disjoint nano open sets V_1 and W_1 in U such that $x \in V_1$ and $f^{-1}(F) \subseteq W_1$. Since f is nano open, the images $f(V_1)$ and $f(W_1)$ are nano open sets in $(V, \tau_R(Y))$. Moreover, because f is injective, $f(V_1) \cap f(W_1) = \emptyset$. We now show that $F \subseteq f(W_1)$. Let $z \in F$. Then $f^{-1}(z) \in f^{-1}(F) \subseteq W_1$, and hence $z = f(f^{-1}(z)) \in f(W_1)$. Thus, $F \subseteq f(W_1)$. Therefore, we have two disjoint nano open sets $f(V_1)$ and $f(W_1)$ in $(V, \tau_R(Y))$ such that $y \in f(V_1)$ and $F \subseteq f(W_1)$. This shows that $(V, \tau_R(Y))$ is nano αg^*p-regular. Hence, the theorem is proved.

Theorem 3.9: Let $f: (U, \tau_R(X)) \longrightarrow (V, \tau_R(Y))$ be an injective function which is nano αg^*p-continuous and nano closed. If $(V, \tau_R(Y))$ is a nano αg^*p-regular space, then $(U, \tau_R(X))$ is also a nano αg^*p-regular space.

Proof: Let $f:(U, \tau_R(X)) \longrightarrow (V, \tau_R(Y))$ be an injective function that is nano αg^*p-continuous and nano closed, and suppose that $(V, \tau_R(Y))$is a nano αg^*p-regular space. We shall prove that $(U, \tau_R(X))$is also nano αg^*p-regular. Let Fbe any nano αg^*p-closed set in $(U, \tau_R(X))$ and let $x \in U \setminus F$. Since f is nano αg^*p-continuous, the image $f(F)$ is a nano αg^*p-closed set in $(V, \tau_R(Y))$. Moreover, $f(x) \notin f(F)$ because f is injective. Since $(V, \tau_R(Y))$ is nano αg^*p-regular, there exist two disjoint nano open sets V_1 and W_1 in V such that $f(x) \in V_1$ and $f(F) \subseteq W_1$. Because $f$is injective, the inverse images $f^{-1}(V_1)$and $f^{-1}(W_1)$are disjoint subsets of U.

Furthermore, f is nano closed, so $f^{-1}(W_1)$ is a nano open set in U. Similarly, by the continuity of f, $f^{-1}(V_1)$ is nano open in U. Clearly, $x \in f^{-1}(V_1)$ and $F \subseteq f^{-1}(W_1)$. Therefore, for the point x and the nano αg^*p-closed set Fin $(U, \tau_R(X))$, we have found two disjoint nano open sets separating them. This proves that $(U, \tau_R(X))$ is nano αg^*p-regular. Hence, the theorem is proved.

Theorem 3.10: Let $f:(U, \tau_R(X)) \longrightarrow (V, \tau_R(Y))$ be an injective function which is nano αg^*p-irresolute and nano closed. If $(V, \tau_R(Y))$ is a nano αg^*p-regular space, then $(U, \tau_R(X))$ is also a nano αg^*p-regular space.

4. Nano *αg*p*-Normal Spaces

Definition 4.1: A nano topological space $(U, \tau_R)$is said to be nano αg^*p-normal if for every pair of disjoint nano αg^*p-closed sets F and G in U, there exist two disjoint nano open sets $V$and Win Usuch that$F \subseteq V$and$G \subseteq W$.

Theorem 4.2: Every nano normal space is nano αg^*p-normal space.

Proof: Let (U, τ_R)be a nano normal space. To show that $U$is nano αg^*p-normal, let $A$and Bbe any two disjoint nano αg^*p-closed sets in U. By the definition of nano αg^*p-closed sets, there exist nano closed sets $F_1$and F_2such that $A \subseteq F_1$and $B \subseteq F_2$. Since $A \cap B = \emptyset$, we may choose $F_1$and $F_2$to be disjoint. As $U$is nano normal, there exist disjoint nano open sets G and $H$with $F_1 \subseteq G$and $F_2 \subseteq H$. Consequently, $A \subseteq G$and $B \subseteq H$with $G \cap H = \emptyset$. Hence, every pair of disjoint nano αg^*p-closed sets can be separated by disjoint nano open sets, and therefore $U$is nano αg^*p-normal.

Remark 4.3: The reverse implication of the above Theorem need not be true can be seen from the following example.

For example, Let $U = \{a, b, c\}$ with nano topology $\tau_R = \{\emptyset, \{a\}, \{a, b\}, U\}$. The nano closed sets are $\emptyset, \{c\}, \{b, c\}, U$. The disjoint nano closed sets $\{c\}$and $\{b, c\}$cannot be separated by disjoint nano open sets; hence the space is not nano normal. However, every pair of disjoint nano αg^*p-closed sets can be separated by nano open sets, so the space is nano αg^*p-normal. Thus, the converse does not hold.

Theorem 4.4 Let Ube a nano topological space. The following conditions are equivalent:

i. $U$is a nano αg^*p-normal space.
ii. For each nano αg^*p-closed set $A$and each nano αg^*p-open set V with $A \subseteq V$, there exists a nano open set Wsuch that $A \subseteq W \subseteq \mathrm{pCl}(W) \subseteq V$.
iii. For each pair of distinct nano αg^*p-closed sets $A$and B, there exists a nano open set Wsuch that $A \subseteq W$and$\mathrm{pCl}(W) \cap B = \emptyset$.
iv. For each pair of distinct nano αg^*p-closed sets $A$and B, there exist nano open sets $V$and W such that $A \subseteq V, B \subseteq W$,and$\mathrm{pCl}(V) \cap \mathrm{pCl}(W) = \emptyset$.

Proof: (i) ⇒ (ii): Assume that $U$is a nano αg^*p-normal space. Let Abe a nano αg^*p-closed set and let Vbe a nano αg^*p-open set such that $A \subseteq V$. Then $U \setminus V$ is a nano αg^*p-closed set and A and $U \setminus V$ are disjoint. By nano αg^*p-normality, there exist disjoint nano open sets $W$and Hsuch that $A \subseteq W$and $U \setminus V \subseteq H$. Since $W \cap H = \emptyset$, we have $\mathrm{pCl}(W) \subseteq U \setminus H \subseteq V$. Hence, $A \subseteq W \subseteq \mathrm{pCl}(W) \subseteq V$.

(ii) ⇒ (iii): Let $A$and Bbe two distinct nano αg^*p-closed sets. Then $B \subseteq U \setminus A$, where $U \setminus A$is a nano αg^*p-open set. By (ii), there exists a nano open set W such that $B \subseteq W \subseteq \mathrm{pCl}(W) \subseteq U \setminus A$. This implies $\mathrm{pCl}(W) \cap A = \emptyset$. Renaming sets if necessary, we obtain a nano open set Wsuch that $A \subseteq W$and $\mathrm{pCl}(W) \cap B = \emptyset$.

(iii) ⇒ (iv): Let $A$and Bbe two distinct nano αg^*p-closed sets. By (iii), there exists a nano open set Vsuch that $A \subseteq V$and $\mathrm{pCl}(V) \cap B = \emptyset$. Since $\mathrm{pCl}(V)$is nano closed and disjoint from B, applying (iii) again, there exists a nano open set Wsuch that $B \subseteq W$and $\mathrm{pCl}(W) \cap \mathrm{pCl}(V) = \emptyset$.

(iv) ⇒ (i): Let $A$and Bbe any two disjoint nano αg^*p-closed sets in U. By (iv), there exist nano open sets $V$and Wsuch that $A \subseteq V$, $B \subseteq W$, and $\mathrm{pCl}(V) \cap \mathrm{pCl}(W) = \emptyset$.
In particular, $V \cap W = \emptyset$. Hence, A and B can be separated by disjoint nano open sets, and therefore $U$is a nano αg^*p-normal space.

Theorem 4.5: Let U be a nano topological space. Then $U$is a nano αg^*p-normal space if and only if for any two disjoint nano αg^*p-closed sets A and Bin U, there exist nano open sets $V$and Win Usuch that $A \subseteq V, B \subseteq W$ and $\mathrm{pCl}(V) \cap \mathrm{pCl}(W) = \emptyset$.

Proof: The proof follows from the Theorem 4.5.

Theorem 4.6: Every nano αg^*p-closed subspace of a nano αg^*p-normal space is nano αg^*p-normal space.

Proof: Let U be a nano αg^*p-normal space and let Y be a nano αg^*p-closed subspace of U. We shall prove that Y, with the subspace nano topology, is also a nano αg^*p-normal space. Let A and B be two disjoint nano αg^*p-closed sets in the subspace Y. By the definition of the subspace topology, there exist nano αg^*p-closed sets A_1 and B_1 in U such that $A = A_1 \cap Y$ and $B = B_1 \cap Y$. Since A and B are disjoint, $A_1 \cap B_1 \cap Y = \emptyset$. Because Y is nano αg^*p-closed in U, this implies that $A_1 \cap B_1 = \emptyset$. Since U is nano αg^*p-normal, there exist nano open sets V and W in U such that $A_1 \subseteq V, B_1 \subseteq W$ and $\mathrm{pCl}(V) \cap \mathrm{pCl}(W) = \emptyset$. Now consider the sets $V \cap Y$ and $W \cap Y$. These are nano open sets in the subspace Y, and they satisfy $A \subseteq V \cap Y, B \subseteq W \cap Y$. Moreover, $\mathrm{pCl}_Y(V \cap Y) \subseteq \mathrm{pCl}(V) \cap Y$ and $\mathrm{pCl}_Y(W \cap Y) \subseteq \mathrm{pCl}(W) \cap Y$. Hence, $\mathrm{pCl}_Y(V \cap Y) \cap \mathrm{pCl}_Y(W \cap Y) = \emptyset$. Thus, the disjoint nano αg^*p-closed sets A and B in Y can be separated by disjoint nano open sets in Y. Therefore, Y is a nano αg^*p-normal space.

Theorem 4.7: Let $f: (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a bijective function. If f is nano open and nano αg^*p-irresolute, and if $(U, \tau_R(X))$ is a nano αg^*p-normal space, then $(V, \tau_R(Y))$ is also a nano αg^*p-normal space.

Proof: Let $f: (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a bijective function which is nano open and nano αg^*p-irresolute, and suppose that $(U, \tau_R(X))$ is a nano αg^*p-normal space. We show that $(V, \tau_R(Y))$ is also nano αg^*p-normal. Let A and B be any two disjoint nano αg^*p-closed sets in V. Since f is nano αg^*p-irresolute, the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are nano αg^*p-closed sets in U. Because f is bijective, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint. As $(U, \tau_R(X))$ is nano αg^*p-normal, there exist nano open sets V_1 and W_1 in U such that $f^{-1}(A) \subseteq V_1, f^{-1}(B) \subseteq W_1$, and $\mathrm{pCl}(V_1) \cap \mathrm{pCl}(W_1) = \emptyset$. Since f is nano open, the images $f(V_1)$ and $f(W_1)$ are nano open sets in V. Moreover, by the bijectivity of f, $A \subseteq f(V_1)$ and $B \subseteq f(W_1)$. Now suppose, to the contrary, that $\mathrm{pCl}(f(V_1)) \cap \mathrm{pCl}(f(W_1)) \neq \emptyset$. Then there exists a point $y \in V$ belonging to both closures. Since f is bijective, there exists $x \in U$ such that $f(x) = y$. Using the nano αg^*p-irresolute nature of f, this would imply $x \in \mathrm{pCl}(V_1) \cap \mathrm{pCl}(W_1)$, which contradicts the fact that $\mathrm{pCl}(V_1) \cap \mathrm{pCl}(W_1) = \emptyset$. Hence, $\mathrm{pCl}(f(V_1)) \cap \mathrm{pCl}(f(W_1)) = \emptyset$. Thus, the disjoint nano αg^*p-closed sets A and B in V can be separated by nano open sets $f(V_1)$ and $f(W_1)$. Therefore, $(V, \tau_R(Y))$ is a nano αg^*p-normal space.

Theorem 4.8: Let $f: (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a function. If f is strongly nano αg^*p-open, continuous, and almost nano αg^*p-irresolute, and if $(U, \tau_R(X))$ is a nano αg^*p-normal space, then $(V, \tau_R(Y))$ is also a nano αg^*p-normal space.

Proof: Let $f: (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a function which is strongly nano αg^*p-open, continuous, and almost nano αg^*p-irresolute, and suppose that $(U, \tau_R(X))$ is a nano αg^*p-normal space. We shall prove that $(V, \tau_R(Y))$ is also nano αg^*p-normal. Let A and B be any two disjoint nano αg^*p-closed sets in V. Since f is almost nano αg^*p-irresolute, the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are nano αg^*p-closed sets in U. Clearly, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint. Because $(U, \tau_R(X))$ is nano αg^*p-normal, there exist nano open sets V_1 and W_1 in U such that $f^{-1}(A) \subseteq V_1, f^{-1}(B) \subseteq W_1$, and $\mathrm{pCl}(V_1) \cap \mathrm{pCl}(W_1) = \emptyset$. Since f is strongly nano αg^*p-open, the images $f(V_1)$ and $f(W_1)$ are nano open sets in V. Moreover, by continuity of f, $\mathrm{pCl}(f(V_1)) \subseteq f(\mathrm{pCl}(V_1))$ and $\mathrm{pCl}(f(W_1)) \subseteq f(\mathrm{pCl}(W_1))$. Since $\mathrm{pCl}(V_1) \cap \mathrm{pCl}(W_1) = \emptyset$, it follows that $\mathrm{pCl}(f(V_1)) \cap \mathrm{pCl}(f(W_1)) = \emptyset$. Also, from the definition of inverse images, $A \subseteq f(V_1)$ and $B \subseteq f(W_1)$. Thus, the disjoint nano αg^*p-closed sets A and B in V are contained in disjoint nano open sets whose nano closures are also disjoint. Hence, $(V, \tau_R(Y))$ satisfies the definition of a nano αg^*p-normal space.

Conclusion: This paper introduced the concepts of nano αg^*p-regular and nano αg^*p-normal spaces as generalized separation axioms in nano topological spaces. Several characterizations were established, and their relationships with nano regular and nano normal spaces were examined, showing that the classical notions imply the generalized ones, while the converses do not hold in general. Preservation properties under subspaces and mappings were also investigated. These results extend the framework of nano topology and provide a basis for further studies on generalized separation axioms.

References

1. Bhuvaneswari, K., & Mythili Ganapriya, K. (2014). Nano generalized closed sets in nano topological spaces. *International Journal of Scientific and Research Publications*, 4(5). https://www.ijsrp.org/research-journal-0514.php
2. Hussein, G. A., & Al-Nafie, Z. D. (2018). Regularity and normality of nano topological spaces modeled by pre-open set structures. *Journal of Applied Science and Computations*, 5(10), 431–438. https://doi.org/10.47974/JIM-1738
3. Pandiammal, C., & Deepika, K. (2019). Nano generalized regular continuity in nano

topological space. *International Journal of Mathematics Trends and Technology*, 65(3), 1–6. https://doi.org/10.14445/22315373/IJMTT-V65I3P527
4. Pawlak, Z. (1982). Rough sets. *International Journal of Computer and Information Sciences*,11, 341–356. https://doi.org/10.1007/BF01001956
5. Reilly, I. L., & Vamanamurty, M. K. (1985). On α-sets in topological spaces. *Tamkang Journal of Mathematics*, 16, 7–11. https://doi.org/10.1007/BF01955019
6. Sathishmohan, P., Rajasekar, S., & Kalaiarasi, R. (2019). More on nano semi pre-regular and nano semi pre-normal spaces. *Malaya Journal of Matematik*, S(1), 367–374. https://doi.org/10.26637/MJM0S01/0066
7. Thivagar, M. L. (2012). Nano topological spaces. In *Proceedings of the International Conference on Mathematical Computer Engineering* (Vol. 1, pp. 1–5). https://surl.li/pvwddr
8. Thivagar, M. L., & Richard, C. (2013). On nano continuity. *Mathematical Theory and Modeling*, 3(7), 32–37. https://doi.org/10.7176/MTM,
9. Thivagar, M. L., & Richard, C. (2013). On nano forms of weakly open sets. *International Journal of Mathematics and Statistics Invention*, 1(1), 31–37. https://surl.li/amqrsv
10. Thivagar, M. L., Jafari, S., & Sutha Devi, V. (2017). On new class of contra continuity in nano topology. *Italian Journal of Pure and Applied Mathematics*, (37), 1–10. https://surl.li/vmeevt
11. Upadhya, A. C. (2017). On quasi nano *p*-normal spaces. *International Journal of Recent Scientific Research*, 8(6), 17748–17751. http://dx.doi.org/10.24327/ijrsr.2017.0806.0415
12. Upadhya, A. C., & Mamata, M. K. (2021). On nano *p*-regular and nano *p*-normal spaces. *International Journal of Mathematics and Its Applications*, 9(3), 161–163. https://ijmaa.in/index.php/ijmaa/article/view/56
13. Upadhya, A. C. (2025). Nano *g**α-normal spaces in nano topological spaces. *Kongunadu Research Journal*, 12(1), 18–22. https://doi.org/10.26524/krj.2025.4,